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Bayesian and Non Bayesian Parameter Estimation for Bivariate Pareto Distribution Based on Censored Samples

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ABSTRACT
This paper deals with Bayesian and non-Bayesian methods for estimating parameters of the bivariate Pareto (BP) distribution based on censored samples are considered with shape parameters λ and known scale parameter β. The maximum likelihood estimators MLE of the unknown parameters are derived. The Bayes estimators are obtained with respect to the squared error loss function and the prior distributions allow for prior dependence among the components of the parameter vector. .Posterior distributions for parameters of interest are derived and their properties are described**.** If the scale parameter is known, the Bayes estimators of the unknown parameters can be obtained in explicit forms under the assumptions of independent priors. An extensive computer simulation is used to compare the performance of the proposed estimators using MathCAD (14).
Keywords- bivariate Pareto distribution, censored samples, importance sampling, maximum likelihood estimators, prior distribution and posterior analysis.

I. Introduction

The censoring time (T) is assumed to be independent of the life times (X, Y) of the two components. The bivariate density function of (X, Y) is denoted by $f_{X,Y}(x,y)$. The considered situation occurs for example in medical studies of paired organs like kidneys, eyes, lungs, or any other paired organs of an individual as a two components system which works under interdependency circumstances. Failure of an individual may censor failure of either one of the paired organ or both. This scheme of censoring is right censoring.

There is similar situation in engineering science whenever sub-systems are considered having two components with life times (X, Y) being independent of the life time (T) of the entire system. However, failure of the main system may censor failure of either one component or both. [See, Hanagal and Ahmadi [1]]

Censoring may also occur in other ways. Patients may be lost to follow up during the study, the patient may decide to move elsewhere therefore the experimenter may not follow him or her again, or the patients may become non-cooperative which is due to some bad side effects of the therapy. Such cases are called withdrawal from the study. A patient with censored data contributes valuable information and should therefore not be omitted from the analysis. Hanagal [2, 3] derived maximum likelihood estimators of the parameters for the case of univariate right censoring.

The rest of the paper is organized as follows. In Section 2, the bivariate Pareto distribution is introduced, the estimation of bivariate Pareto distribution based on censored samples is proposed in Section 3. Section 4 discussed the Bayesian parameters estimation for Pareto distribution based on censored samples. The maximum likelihood estimates (MLEs) of the parameters of the bivariate Pareto of Marshall-Olkin are obtained based on censored samples in Section 5. Finally, simulation results and conclusions are laid out in Section 6.

II. The bivariate Pareto distribution

The Pareto distribution was first proposed as a model for the distribution of incomes, it is also used as a model for the distribution of city populations within a given area. [See,Johnson andKotz [4]].
The probability distribution function and the cumulative distribution functions are defined respectively by the following functions:

$$f(x,\beta,\lambda)=\frac{\lambda}{\beta}\left(\frac{x}{\beta}\right)^{-\lambda-1} \qquad ,x>\beta \quad ,\lambda,\beta>0 \tag{1}$$

$$F(x,\beta,\lambda)=1-\left(\frac{x}{\beta}\right)^{-\lambda} \qquad ,x>\beta \quad ,\lambda,\beta>0 \tag{2}$$

Veenus and Nair[5] proposed a bivariate Pareto (BP) distribution with many interesting properties like marginal Pareto, bivariate loss of memory property and they proposed the survival function for $x, y > 0$, $\lambda_1 > 0, \lambda_2 > 0$ and $\lambda_3 > 0$ as follows:

$$\overline{F}(x,y)=\left(\frac{x}{\beta}\right)^{-\lambda_1}\left(\frac{y}{\beta}\right)^{-\lambda_2}\left(\frac{\max(x,y)}{\beta}\right)^{-\lambda_3}, \qquad x,y>\beta>0 \tag{3}$$

Where

$\lambda = \lambda_1 + \lambda_2 + \lambda_3$

Also they proposed the joint probability density function $f_{X,Y}(x,y)$ of X and Y as follows:

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{(\lambda_2+\lambda_3)\lambda_1}{\beta^2}\left(\dfrac{x}{\beta}\right)^{-\lambda_1-1}\left(\dfrac{y}{\beta}\right)^{-(\lambda_2+\lambda_3)-1} & if \quad \beta < x < y < \infty \\ \dfrac{(\lambda_1+\lambda_3)\lambda_2}{\beta^2}\left(\dfrac{x}{\beta}\right)^{-(\lambda_1+\lambda_3)-1}\left(\dfrac{y}{\beta}\right)^{-\lambda_2-1} & if \quad \beta < y < x < \infty \\ \dfrac{\lambda_3}{\beta}\left(\dfrac{x}{\beta}\right)^{-\lambda-1} & if \quad \beta < x = y < \infty \end{cases} \quad (4)$$

Where

$\lambda = \lambda_1 + \lambda_2 + \lambda_3$

III. Estimation for BP Distribution Based on Censored Samples

The univariate random censoring scheme given by Hanagal [2] is used for estimating the bivariate life time distribution, which takes into account that individuals do not enter at the same time the study and a withdrawal of an individual will censor both life times of the components which in the sequel will be called implants, because the model was developed and applied in the framework of teeth implants for upper and lower jaws.

Suppose that there are n independent pairs of implants under study, where the i[th]pair of implants have life times (x_i, y_i) and a censoring time (t_i).

Let the censored random life of the i[th] pair be denoted by (x_i, y_i) .

Then (x_i, y_i) are defined as follows:

$$(x_i, y_i) = \begin{cases} (x_i, y_i) & \text{if} \quad \max(x_i, y_i) < t_i \\ (x_i, t_i) & \text{if} \quad x_i < t_i < y_i \\ (t_i, y_i) & \text{if} \quad y_i < t_i < x_i \\ (t_i, t_i) & \text{if} \quad \min(x_i, y_i) > t_i \end{cases} \quad (5)$$

There are six different types of events which might occur with respect to $(x_i, y_i), i = 1, \cdots, n$. These are the following:

1. Type 1: $x_i < y_i < t_i$
2. Type 2: $y_i < x_i < t_i$
3. Type 3: $x_i = y_i < t_i$
4. Type 4: $x_i < t_i < y$
5. Type 5: $y_i < t_i < x_i$
6. Type 6: $t_i < \min(x_i, y_i)$

Let n_1, n_2, n_3, n_4, n_5 and n_6 be the numbers of observations representing the differenttypes of events with $n = n_1 + n_2 + n_3 + n_4 + n_5 + n_6$. Then the likelihood function L for a sample $((x_1, y_1), \cdots, (x_n, y_n))$is given as follows:

$$L = \left(\prod_{i=1}^{n_1} f_1(x_i, y_i)\overline{G}(t_i)\right)\left(\prod_{i=1}^{n_2} f_2(x_i, y_i)\overline{G}(t_i)\right)\left(\prod_{i=1}^{n_3} f_3(x_i, y_i)\overline{G}(t_i)\right)$$
$$\left(\prod_{i=1}^{n_4} f_4(x_i, t_i)g(t_i)\right)\left(\prod_{i=1}^{n_5} f_5(t_i, y_i)g(t_i)\right)\left(\prod_{i=1}^{n_6} \overline{F}(t_i, t_i)g(t_i)\right) \quad (6)$$

Where

$$g(t_i, \theta) = \theta\, e^{-\theta(t_i - \beta)} \qquad t_i > \beta \;\; , \; \theta, \beta > 0$$
$$\overline{G}(t_i) = P(T_i > \max(x_i, \mathrm{y_i})) = e^{-\theta[\max(x_i, \mathrm{y_i}) - \beta]}$$
$$\text{where} \qquad \max(x_i, \mathrm{y_i}) > \beta \;\; , \; \theta > 0$$
$$f_1(x_i, y_i) = \frac{(\lambda_2 + \lambda_3)\lambda_1}{\beta^2}\left(\frac{x_i}{\beta}\right)^{-\lambda_1-1}\left(\frac{y_i}{\beta}\right)^{-(\lambda_2+\lambda_3)-1}$$
$$f_2(x_i, y_i) = \frac{(\lambda_2 + \lambda_3)\lambda_1}{\beta^2}\left(\frac{x_i}{\beta}\right)^{-(\lambda_1+\lambda_3)-1}\left(\frac{y_i}{\beta}\right)^{-\lambda_2-1}$$
$$f_3(x_i) = \frac{\lambda_3}{\beta}\left(\frac{x_i}{\beta}\right)^{-\lambda-1}$$
$$f_4(x_i, t_i) = \lim_{\Delta x \to 0}\frac{P(x < X < x + \Delta x | y > t)P(Y > t)}{\Delta x}$$
$$= \frac{\lambda_1}{\beta^2}\left(\frac{x_i}{\beta}\right)^{-\lambda_1-1}\left(\frac{t_i}{\beta}\right)^{-(\lambda_2+\lambda_3)}$$
$$f_5(t_i, y_i) = \lim_{\Delta y \to 0}\frac{P(y < Y < y + \Delta y | x > t)P(x > t)}{\Delta y}$$
$$= \frac{\lambda_1}{\beta^2}\left(\frac{y_i}{\beta}\right)^{-\lambda_2-1}\left(\frac{t_i}{\beta}\right)^{-(\lambda_1+\lambda_3)}$$
$$\overline{F}(t_i, t_i) = P(X_i > t_i, Y > t_i) = \frac{\lambda_3}{\beta}\left(\frac{t_i}{\beta}\right)^{-\lambda}$$

Then the log - likelihood function L for a sample $((x_1, y_1), \cdots, (x_n, y_n))$is given by:

$$\ln L = \sum_{i=1}^{n_1}\ln\left[f_1(x_i, y_i)\overline{G}(t_i)\right] + \sum_{i=1}^{n_2}\ln\left[f_2(x_i, y_i)\overline{G}(t_i)\right] + \sum_{i=1}^{n_3}\ln\left[f_3(x_i, y_i)\overline{G}(t_i)\right]$$
$$+ \sum_{i=1}^{n_4}\ln\left[f_4(x_i, t_i)g(t_i)\right] + \sum_{i=1}^{n_5}\ln\left[f_5(t_i, y_i)g(t_i)\right] + \sum_{i=1}^{n_6}\ln\left[\overline{F}(t_i, t_i)g(t_i)\right]$$
$$= \sum_{i=1}^{n_1}\ln\left[f_1(x_i, y_i)\right] + \sum_{i=1}^{n_2}\ln\left[f_2(x_i, y_i)\right] + \sum_{i=1}^{n_3}\ln\left[f_3(x_i, y_i)\right]$$
$$+ \sum_{i=1}^{n_4}\ln\left[f_4(x_i, t_i)\right] + \sum_{i=1}^{n_5}\ln\left[f_5(t_i, y_i)\right] + \sum_{i=1}^{n_6}\ln\left[\overline{F}(t_i, t_i)\right]$$
$$+ \sum_{i \in A}\ln\left[\overline{G}(t_i)\right] + \sum_{i \in B}\ln\left[g(t_i)\right] \quad (7)$$

Where

$$\sum_{i \in A}\ln\left[\overline{G}(t_i)\right] = (n_1 + n_2 + n_3)\theta\beta - \theta\sum_{i \in A}\max(x_i, y_i)$$
$$\sum_{i \in B}\ln\left[g(t_i)\right] = (n_4 + n_5 + n_6)\ln(\theta) + (n_4 + n_5 + n_6)\theta\beta - \theta\sum_{i \in B}t_i$$

$$\sum_{i=1}^{n_1}\ln\left[f_1(x_i,y_i)\right]=n_1\left(\ln(\lambda_2+\lambda_3)+\ln(\lambda_1)+(\lambda)\ln(\beta)\right)-(\lambda_1+1)\sum_{i=1}^{n_1}\ln x_i-(\lambda_2+\lambda_3+1)\sum_{i=1}^{n_1}\ln y_i$$

$$\sum_{i=1}^{n_2}\ln\left[f_2(x_i,y_i)\right]=n_2\left(\ln(\lambda_1+\lambda_3)+\ln(\lambda_2)+(\lambda)\ln(\beta)\right)-(\lambda_2+1)\sum_{i=1}^{n_2}\ln y_i-(\lambda_1+\lambda_3+1)\sum_{i=1}^{n_2}\ln x_i$$

$$\sum_{i=1}^{n_3}\ln\left[f_3(x_i,y_i)\right]=n_3\left(\ln(\lambda_3)+(\lambda)\ln(\beta)\right)-(\lambda+1)\sum_{i=1}^{n_3}\ln x_i$$

$$\sum_{i=1}^{n_4}\ln\left[f_4(x_i,t_i)\right]=n_4\left(\ln(\lambda_1)+(\lambda)\ln(\beta)\right)-(\lambda_1+1)\sum_{i=1}^{n_4}\ln x_i-(\lambda_2+\lambda_3)\sum_{i=1}^{n_4}\ln t_i$$

$$\sum_{i=1}^{n_5}\ln\left[f_5(t_i,y_i)\right]=n_5\left(\ln(\lambda_2)+(\lambda)\ln(\beta)\right)-(\lambda_2+1)\sum_{i=1}^{n_5}\ln y_i-(\lambda_1+\lambda_3)\sum_{i=1}^{n_5}\ln t_i$$

$$\sum_{i=1}^{n_6}\ln\left[\overline{F}(t_i,t_i)\right]=n_6(\lambda)\left(\ln(\beta)\right)-(\lambda)\sum_{i=1}^{n_6}\ln t_i$$

Suppose the scale parameter β is known then,

$$\ln L = (n_2+n_5)\ln(\lambda_2)+(n_1+n_4)\ln(\lambda_1)+n_1\ln(\lambda_2+\lambda_3)+n_3\ln(\lambda_3) + n_2\ln(\lambda_1+\lambda_3)+(n_4+n_5+n_6)\ln(\theta)-\lambda_1k_1-\lambda_2k_2-\lambda_3k_3-\theta k_4 - \Psi(x_i,y_i,t_i) \quad (8)$$

Where

$$\Psi(x_i,y_i,t_i)=\sum_{i=1}^{n_1}\ln x_i+\sum_{i=1}^{n_1}\ln y_i+\sum_{i=1}^{n_2}\ln x_i+\sum_{i=1}^{n_2}\ln y_i+\sum_{i=1}^{n_3}\ln x_i+\sum_{i=1}^{n_4}\ln x_i+\sum_{i=1}^{n_5}\ln y_i$$

$$k_1=\sum_{i=1}^{n_1}\ln x_i+\sum_{i=1}^{n_2}\ln x_i+\sum_{i=1}^{n_3}\ln x_i+\sum_{i=1}^{n_4}\ln x_i+\sum_{i=1}^{n_5}\ln t_i+\sum_{i=1}^{n_6}\ln t_i-n\ln(\beta)$$

$$k_2=\sum_{i=1}^{n_1}\ln y_i+\sum_{i=1}^{n_2}\ln y_i+\sum_{i=1}^{n_5}\ln y_i+\sum_{i=1}^{n_4}\ln t_i+\sum_{i=1}^{n_6}\ln t_i-n\ln(\beta)$$

$$k_3=\sum_{i=1}^{n_1}\ln y_i+\sum_{i=1}^{n_2}\ln x_i+\sum_{i=1}^{n_3}\ln x_i+\sum_{i=1}^{n_4}\ln t_i+\sum_{i=1}^{n_5}\ln t_i+\sum_{i=1}^{n_6}\ln t_i-n\ln(\beta)$$

$$k_4=\left[\sum_{i\in A}\max(x_i,y_i)+\sum_{i\in B}t_i-n\beta\right]$$

IV. Bayesian Parameter Estimation for BP Distribution Based on Censored Samples

This Section deals with the Bayesian estimate of BP estimators based on censored samples when the scale parameter β is known; let the same conjugate prior on λ_1,λ_2 and λ_3 is given as follow.

$$\pi(\lambda_r)\propto\lambda_r^{\tau_r-1}\exp(-b_r\lambda_r) \qquad ,r=1,2,3 \quad (9)$$

and conjugate prior for θ is

$$\pi(\theta)\propto\theta^{\tau_4-1}\exp(-b_4\theta) \qquad (10)$$

where λ_1, λ_2,λ_3 and θ have independent gamma priors.

We can rewrite the likelihood equation from equation (8) as follow

$$L=e^{\ln L} = (\lambda_2)^{n_2+n_5}(\lambda_1)^{n_1+n_4}(\lambda_3)^{n_3}(\lambda_2+\lambda_3)^{n_1}(\lambda_1+\lambda_3)^{n_2} (\theta)^{n_4+n_5+n_6}Exp\left(-\lambda_1k_1-\lambda_2k_2-\lambda_3k_3-\theta k_4\right)e^{-\Psi(x_i,y_i,t_i)}$$

Then

$$L\propto\sum_{l=0}^{n_1}\sum_{j=0}^{n_2}\binom{n_1}{l}\binom{n_2}{j}(\lambda_2)^{n_2+n_5+j}(\lambda_1)^{n_1+n_4+l}(\lambda_3)^{n_1+n_2+n_3-l-j}(\theta)^{n_4+n_5+n_6} Exp\left(-\lambda_1k_1-\lambda_2k_2-\lambda_3k_3-\theta k_4\right) \quad (11)$$

The joint posterior density of $\lambda_1,\lambda_2,\lambda_3$ and θ will be :

$$\pi(\lambda_1,\lambda_2,\lambda_3,\theta\mid \underline{x}\ \underline{y})\propto[\sum_{l=0}^{n_1}\sum_{j=0}^{n_2}\binom{n_1}{l}\binom{n_2}{j}(\lambda_1)^{n_1+n_4+\tau_1+l} Exp\left(-[k_1+b_1]\lambda_1\right)(\lambda_2)^{n_2+n_5+\tau_2+j-1} Exp\left(-[k_2+b_2]\lambda_2\right)(\lambda_3)^{n_1+n_2+n_3+\tau_3-l-j} Exp\left(-[k_2+b_2]\lambda_2\right)(\theta)^{n_4+n_5+n_6+\tau_4-1}Exp\left(-[k_4+b_4]\theta\right)]$$

then

$$\pi(\lambda_1,\lambda_2,\lambda_3,\theta\mid \underline{x}\ \underline{y})=C\sum_{l=0}^{n_1}\sum_{j=0}^{n_2}\binom{n_1}{l}\binom{n_2}{j}(\lambda_1)^{a_{1l}-1}Exp\left(-[k_1+b_1]\lambda_1\right)(\lambda_2)^{a_{2j}-1} Exp\left(-[k_2+b_2]\lambda_2\right)(\lambda_3)^{a_{3lj}-1} Exp\left(-[k_2+b_2]\lambda_2\right)(\theta)^{a_4-1}Exp\left(-[k_4+b_4]\theta\right) \quad (12)$$

$$c_{lj}=\binom{n_1}{l}\binom{n_2}{j}\frac{\Gamma a_{1l}}{(k_1+b_1)^{a_{1l}}}\frac{\Gamma a_{2j}}{(k_2+b_2)^{a_{2j}}}\frac{\Gamma a_{3lj}}{(k_3+b_3)^{a_{3lj}}}\frac{\Gamma a_4}{(k_4+b_4)^{a_4}} \quad (13)$$

then

$$C=\frac{1}{\sum_{l=0}^{n_1}\sum_{j=0}^{n_2}c_{lj}} \qquad (14)$$

Where

$a_{1l}=n_1+n_4+\tau_1+l$, $a_{2j}=n_2+n_5+\tau_2+j$, $a_{3lj}=n_1+n_2+n_3+\tau_3+l+j$

and $a_4=n_4+n_5+n_6+\tau_4$

Therefore, under the assumption of independence of λ_1, λ_2,λ_3 and θ, it is possible to get the Bayes estimates of λ_1, λ_2,λ_3 and θ in closed forms, explicitly under the squared error loss function using (12), as follows:

$$\tilde{\lambda}_1=\frac{C}{k_1+b_1}\sum_{l=0}^{n_1}\sum_{j=0}^{n_2}c_{lj}a_{1l} \qquad (15)$$

$$\tilde{\lambda}_2=\frac{C}{k_2+b_2}\sum_{l=0}^{n_1}\sum_{j=0}^{n_2}c_{lj}a_{2j} \qquad (16)$$

$$\tilde{\lambda}_3=\frac{C}{k_3+b_3}\sum_{l=0}^{n_1}\sum_{j=0}^{n_2}c_{lj}a_{3lj} \qquad (17)$$

$$\tilde{\theta}=\frac{C}{k_4+b_4}\sum_{l=0}^{n_1}\sum_{j=0}^{n_2}c_{lj}a_4=\frac{a_4}{k_4+b_4} \qquad (18)$$

V. Non Bayesian Parameter Estimation for BP Distribution Based on Censored Samples

This section deals with MLE of the unknown estimators, it is well known that the closed forms of maximum likelihood estimators of the unknown parameters do not always exist.

From equation (8) take the derivative of the log likelihood $\ln L$ with respect to each parameter set the partial derivatives equal to zero.
Therefore the normal equations are

$$\frac{\partial \ln L}{\partial \lambda_1} = \frac{(n_1+n_4)}{\lambda_1} + \frac{n_2}{\lambda_1+\lambda_3} - k_1 = 0 \qquad (19)$$

$$\frac{\partial \ln L}{\partial \lambda_2} = \frac{(n_2+n_5)}{\lambda_2} + \frac{n_1}{\lambda_2+\lambda_3} - k_2 = 0 \qquad (20)$$

$$\frac{\partial \ln L}{\partial \lambda_3} = \frac{n_3}{\lambda_3} + \frac{n_2}{\lambda_1+\lambda_3} + \frac{n_1}{\lambda_2+\lambda_3} - k_3 = 0 \qquad (21)$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n_4+n_5+n_6}{\theta} - k_4 = 0 \qquad (22)$$

$$\therefore \hat{\theta} = \frac{n_4+n_5+n_6}{k_4} \qquad (23)$$

The likelihood equations (19), (20) and (21) may be solved by a Newton-Raphson procedure, where thesecond order partial derivatives of the log-likelihood function are given by:

$$\frac{\partial^2 \ln L}{\partial \lambda_1^2} = \frac{-n_2}{(\lambda_1+\lambda_3)^2} - \frac{(n_1+n_4)}{\lambda_1^2}$$

$$\frac{\partial^2 \ln L}{\partial \lambda_1 \partial \lambda_2} = \frac{\partial^2 \ln L}{\partial \lambda_2 \partial \lambda_1} = 0$$

$$\frac{\partial^2 \ln L}{\partial \lambda_1 \lambda_3} = \frac{\partial^2 \ln L}{\partial \lambda_3 \lambda_1} = \frac{-n_2}{(\lambda_1+\lambda_3)^2}$$

$$\frac{\partial^2 \ln L}{\partial \lambda_1 \theta} = \frac{\partial^2 \ln L}{\partial \theta \lambda_1} = 0$$

$$\frac{\partial^2 \ln L}{\partial \beta_2^2} = -\frac{(n_2+n_5)}{\lambda_2^2} - \frac{n_1}{(\lambda_2+\lambda_3)^2}$$

$$\frac{\partial^2 \ln L}{\partial \lambda_2 \lambda_3} = \frac{\partial^2 \ln L}{\partial \lambda_3 \lambda_2} = -\frac{n_1}{(\lambda_2+\lambda_3)^2}$$

$$\frac{\partial^2 \ln L}{\partial \lambda_2 \theta} = \frac{\partial^2 \ln L}{\partial \theta \lambda_2} = 0$$

$$\frac{\partial^2 \ln L}{\partial \lambda_3^2} = \frac{-n_3}{\lambda_3^2}$$

$$\frac{\partial^2 \ln L}{\partial \lambda_3 \theta} = \frac{\partial^2 \ln L}{\partial \theta \lambda_3} = 0$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{n_4+n_5+n_6}{\theta^2}$$

The observed Fisher information matrix, I is a (4×4) matrix, where the entries are second order partial derivatives displayed above.

$$I = -\begin{pmatrix} \frac{\partial^2 \ln L}{\partial \lambda_1^2} & \frac{\partial^2 \ln L}{\partial \lambda_1 \partial \lambda_2} & \frac{\partial^2 \ln L}{\partial \lambda_1 \partial \lambda_3} & \frac{\partial^2 \ln L}{\partial \lambda_1 \partial \theta} \\ \frac{\partial^2 \ln L}{\partial \lambda_2 \partial \beta_1} & \frac{\partial^2 \ln L}{\partial \lambda_2^2} & \frac{\partial^2 \ln L}{\partial \lambda_2 \partial \lambda_3} & \frac{\partial^2 \ln L}{\partial \lambda_2 \partial \theta} \\ \frac{\partial^2 \ln L}{\partial \lambda_3 \partial \lambda_1} & \frac{\partial^2 \ln L}{\partial \lambda_3 \partial \lambda_2} & \frac{\partial^2 \ln L}{\partial \lambda_3^2} & \frac{\partial^2 \ln L}{\partial \lambda_3 \partial \theta} \\ \frac{\partial^2 \ln L}{\partial \theta \partial \lambda_1} & \frac{\partial^2 \ln L}{\partial \theta \partial \lambda_2} & \frac{\partial^2 \ln L}{\partial \theta \partial \lambda_3} & \frac{\partial^2 \ln L}{\partial \theta^2} \end{pmatrix} \qquad (24)$$

The inverse of the observed Fisher information matrix is the observed variance-covariance matrix of $\underline{\hat{\theta}} = (\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\theta})'$, the MLEof the parameter $\underline{\theta} = (\lambda_1, \lambda_2, \lambda_3, \theta)'$.

The quantity $\sqrt{n}(\underline{\hat{\theta}} - \underline{\theta})$ has an asymptotic multivariate normal distribution with mean vector zero and observed variance-covariance matrix Σ.

VI. Simulation Study

In this section, an extensive numerical investigation using Mathcad (14) will be carried out to estimate the parameters of the bivariate Pareto distribution based on censored samples. The algorithm for this estimation can be summarized in the following steps:

- **Step(1):** Generate u_i using the Pareto distribution with parameter α_i for $i = 1,2,3$.
- **Step (2):** Let $X = \min(u_1, u_3)$ and $Y = \min(u_2, u_3)$ and, therefore, (X,Y) follows a bivariate Pareto distribution of Marshall-Olkin type.
- **Step (3):** Generate t_i using the two-parameter exponential distribution with parameters θ, β where $t_i s$ are the censoring times.
- **Step (4):** Generate 1000 sets of samples for two cases with respect to the $\lambda_i s$, each set consisted of three samples with sizes n = 20, 35 and 50.
- **Step (5):** The estimates are obtained by taking the mean of the 1000 maximum likelihood estimates and the mean of the 1000 standard deviations from the 1000 samples of size n = 20, 35, and 50. The estimates of the standard deviation of the maximum likelihood estimates of $(\lambda_1, \lambda_2, \lambda_3, \theta)$ are obtained by taking square

root of the diagonal elements of the inverse of the observed Fisher information matrix.

- **Step (6):** The Bayes estimates of λ_1, λ_2, λ_3 and θ are computed based on squared error loss function using equations 15, 16, 17 and 18.
- **Step (7):** The squared deviations are computed.
- **Step (8):** The estimated risk (ER) of the Bayes estimate is obtained.

VII. Conclusion

Simulation results for the corresponding maximum likelihood estimates and the Bayes estimates are summarized in Tables 1 and 2. From these Tables, the following conclusions can be observed on the properties of estimated parameters:

It has been observed that there is a direct proportional relationship between MLE estimators' values and β values. The estimators' values move away from the real parameters values as long as the β value increases. In contrast, it has been seen that standard errors has an indirect proportional relationship with when β value.

Furthermore, the results show that whenever the sample increases the MLE estimators are more close to real values with less standard error, which significantly confirms the consistency property.

Referring to tables (1&2) it is obvious that MLE and Bayesian estimators' values are more close to the real parameter values in case of β =1 unlike when β =2 and the standard error is seen less at β =1 rather than at β =2.

Table (2) explores the Bayesian estimators at different values for the prior distribution parameters ($\tau_1, \tau_2, \tau_3, \tau_4$) and ($, b_1, b_2, b_3, b_4$)and provides the Estimated Risk (ER) depending on the squared error loss function. The Bayesian estimators and ER have been observed to get affected by different values of prior distribution and β .

Additionally it has been seen that Bayesian estimators have a closed form, which it is highly recommended to be gone through and study its properties as a future work.

Table (1) :ML estimators and SE of the point estimate from bivariate Pareto Distribution and 1000 repetitions for different sizes of samples

Parameters		λ_1 1.8	λ_2 1.7	λ_3 1.5	θ 0.3	λ_1 0.8	λ_2 0.6	λ_3 0.9	θ 0.2
β	$n=20$								
1	MLE	1.731	1.545	1.622	0.244	0.712	0.522	0.781	0.156
	SE	0.087	0.122	0.107	0.071	0.089	0.113	0.107	0.079
2	MLE	1.423	1.332	1.243	0.211	0.624	0.456	0.641	0.149
	SE	0.287	0.345	0.432	0.113	0.213	0.296	0.315	0.124
	$n=35$								
1	MLE	1.756	1.612	1. 573	0.267	0.744	0.567	0.823	0.172
	SE	0.066	0.109	0.092	0.054	0.073	0.104	0.098	0.065
2	MLE	1.487	1.384	1.324	0.227	0.635	0.478	0.674	0.158
	SE	0.253	0.299	0.387	0.099	0.198	0.251	0.288	0.107
	$n=50$								
1	MLE	1.783	1.623	1.493	0.279	0.778	0.589	0.887	0.203
	SE	0.064	0.097	0.087	0.049	0.047	0.091	0.076	0.031
2	MLE	1.557	1.427	1.411	0.243	0.654	0.492	0.695	0.173
	SE	0.231	0.267	0.356	0.081	0.141	0.221	0.253	0.082

Table (2) :Bayes estimators (BE) and Estimated Risk (ER) of the point estimate from bivariate Pareto Distribution and 1000 repetitions

Parameters		λ_1 1.8	λ_2 1.7	λ_3 1.5	θ 0.3	λ_1 0.8	λ_2 0.6	λ_3 0.9	θ 0.2
β	$\tau_1 = 1.3, \tau_2 = 1.7, \tau_3 = 1.6, \tau_4 = 1.5$, $b_1 = 0.5, b_2 = 0.4, b_3 = 0.3, b_4 = 0.2$								
1	BE	1.433	1.356	1.358	0.277	0.655	0.498	0.692	0.153
	ER	0.277	0.124	0.147	0.109	0.133	0.243	0.422	0.082
2	BE	1.324	1.311	1.233	0.255	0.627	0.466	0.647	0.147
	ER	0.297	0.139	0.323	0.117	0.218	0.299	0.318	0.135
	$\tau_1 = 0.3, \tau_2 = 0.7, \tau_3 = 0.6, \tau_4 = 0.5$, $b_1 = 1.5, b_2 = 1.4, b_3 = 1.3, b_4 = 1.2$								
1	BE	1.556	1.633	1. 532	0.282	0.678	0.511	0.724	0.174
	ER	0.244	0.111	0.123	0.091	0.121	0.213	0.379	0.073
2	BE	1.471	1.309	1.314	0.243	0.631	0.471	0.681	0.156
	ER	0.314	0.143	0.388	0.123	0.188	0.249	0.287	0.101

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